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Generating Future and Present Values of Annuity Using Interest Rate Theory

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Abstract

In this study, the future and present values of annuity were generated on the basis of interest rate theory. This was anchored on the fact that every compound interest problem involves the annual rate and the rate per compounding period. It was also shown that as the frequency of compounding periods increases, the compound amount behaviour $\left(1+\frac{r}{n}\right)^n$ tends to exponential growth rate $\left(e^r\right)$. It was deduced that effective rate of interest is a function of nominal rates and compounding periods.

Keywords: Annuity future value, Annuity present value, Continuous compounding, Compounding periods, Periodic payments

1. Introduction

A common component to all financial transactions is the investment of money to interest. In order to analyze the financial transaction, a clear understanding of the concept of interest is required (Marcel, 2013; Kellison, 1991; Cairns et al, 2006; Ledlie et al, 2007).

Interest can be viewed from different perspectives. It is the cost of borrowing money for some period of time. However, the amount of interest depends among other things, on the amount of money borrowed and repayment time. Suppose $A(t)$ denotes the amount of value of an investment at time t years. Then the interest earned during the time t years and $t+s$ years is $A_{(t+s)}$ - A_t and the annual interest rate

$$r \quad = \frac{A(t+1)-A(t)}{A(t)} \tag{1}$$

1.1 Generating Compound Amount from the Interest Variables

If the simple interest is denoted by $I = PRT$, where I is the amount of interest earned after T years, P is the amount invested and R is the annual interest rate, then future value of the account after T years denoted by

$$\begin{aligned} F &= P+I \\ &= P+PRT \\ &= P(1+RT) \end{aligned} \tag{2}$$

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If the interest is compounded annually, then the amounts at the end of T years are specified

$$\begin{aligned} A_1 &= P(1+R) \\ A_2 &= A_1 + A_1 R \\ &= A_1(1+R) \\ &= P(1+R)^2 \\ A_3 &= A_2 + A_2 R \\ &= A_2(1+R) \\ &= P(1+R)^2(1+R) \\ &= P(1+R)^3 \\ &. \\ &. \\ &. \\ A_t &= P(1+R)^t \qquad (3) \end{aligned}$$

where t= 0,1,2,... Thus, the building block for the annual compound interest formula is the simple interest formula (Awogbemi, 2012; Lin and Cox, 2005a ; Lin and Cox, 2005b).

Suppose the interest is compounded n times in a year. At the end of every period of time $\frac{1}{n}$ years, the period's interest is added to the principal to earn interest in future periods. Thus, we have divided the year into n intervals, each with duration of $\frac{1}{n}$ years. The simple interest formula is applied to P over each period (sub interval). Thus, for one period, the interest earned is:

$$\begin{aligned} I &= PRT \\ &= P\frac{r}{n} \qquad (4) \end{aligned}$$

The future value after one period $\left(t = \frac{1}{n}\text{years}\right)$ denoted by

$$\begin{aligned} A_1 &= P(1 + r\frac{1}{n}) \\ &= P(1+\frac{r}{n}) \qquad (5) \end{aligned}$$

After two periods:

$$\begin{aligned} \mathrm{A}_2 &= \left[P\left(1+\frac{r}{n}\right)\right]\left(1+\frac{r}{n}\right) \\ &= P\left(1+\frac{r}{n}\right)^2 \qquad (6) \end{aligned}$$

After three periods:

$$A_3 = \left[P\left(1+\frac{r}{n}\right)^2\right]\left(1+\frac{r}{n}\right)$$

$$= P\left(1+\frac{r}{n}\right)^3$$

$$\vdots \qquad (7)$$

$$A_n = P\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\cdots\left(1+\frac{r}{n}\right)$$

$$= P\left(1+\frac{r}{n}\right)^n$$

After n periods, P is compounded n times.

Thus, the amount due in t years if there are n compounding periods per year is computed as:

$$A_t = P\left(1+\frac{r}{n}\right)^{nt},$$

where $\frac{r}{n}$ is the rate per period, r is the annual interest rate, n the number of compounding periods in a year, t is the number of years and nt is the total number of compounding periods in t years.

2. Continuous Compounding of Interest

An interest is continuously compounded if the number of compounding periods increases infinitely. The implication of this is that as the frequency of compounding increases, $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$ tends to e^r with the compound amount $A = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$

Thus, the amount due for continuously compounded interest denoted by $A = Pe^{rt}$ (Buchanan, 2010; Willet, 2004).

2.1 Lemma 1

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n = e \qquad (8)$$

Proof

Suppose of f and g are continuous and differentiable functions on an open interval containing x = a, and that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$

If $\lim_{x\to a}\frac{f^1(x)}{g^1(x)}$ has a finite limit, then $\lim_{x\to a}\frac{f^1(x)}{g^1(x)} = \lim_{x\to a}\frac{f^1(x)}{g^1(x)}$

Let a new variable $k = \frac{1}{n} \Rightarrow n = \frac{1}{k}$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}(1+k)^{\frac{1}{k}} \qquad (9)$$

Introducing a dependent variable $y = (1+k)^{\frac{1}{k}}$ and taking the log of both sides gives

$$\log y = \log\left[(1+k)^{\frac{1}{k}}\right]$$
$$= \frac{1}{k}\log(1+k)$$
$$= \log\frac{(1+k)}{k} \qquad (10)$$

$\lim_{k\to 0}\log y = \lim_{k\to 0}\frac{\log(1+k)}{k}$ (an indeterminate form of the type $\frac{0}{0}$)

By L'Hôpital's Rule, we have

$$\lim_{k\to 0}\log y = \lim_{k\to 0}\frac{\log(1+k)}{k}$$
$$= \lim_{k\to 0}\frac{\frac{1}{(1+k)}}{1}$$
$$= 1 \qquad (11)$$

We have shown that $\log y \to 1$ as $k \to 0$ .The continuity of the exponential function implies that $e^{\log y} \to e^{1}$ as $k \to 0$. This implies that $y \to e$ as $k \to 0$ (Zenou, 2006; McCutcheon and Scott, 1986)

Hence, we have proved that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \lim_{k\to 0}(1+k)^{\frac{1}{k}} = e \qquad (12)$$

2.2 Lemma 2

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt} \qquad (13)$$

Proof

The future value for continuously compounded interest assumes that a limit exists.

Let the limit be L

$$L = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}$$
$$\ln(L) = \ln\left[L = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}\right]$$
$$\ln(L) = \lim_{n\to\infty}\ln\left[P\left(1+\frac{r}{n}\right)^{nt}\right]$$
$$\ln(L) = \lim_{n\to\infty}\left[\ln(P) + \ln\left(1+\frac{r}{n}\right)^{nt}\right]$$
$$\ln(L) = \lim_{n\to\infty}\left[\ln(P) + nt\,\ln\left(1+\frac{r}{n}\right)\right] \qquad (14)$$

But as n gets larger, $\frac{r}{n}$ gets really small. Therefore, log approximation $\ln(1+h) \to h$ is used to get

$$\ln(L) = \lim_{n\to\infty}\left(\ln(P) + nt.\frac{r}{n}\right)$$
$$\ln(L) = \lim_{n\to\infty}(\ln P + rt) = L = Pe^{rt} \qquad (15)$$

3. Annuities

An annuity is a sequence of payments made at regular intervals of time. The time period in which these payments are made is referred to the term of the annuity (Teresa, 2010; Lando, 2004).

An annuity in which payments are made at the end of each payment is called ordinary annuity, where as an annuity in which payments are made at the beginning of each period is called an annuity due (Nasiru and Awogbemi, 2013). An annuity in which the payment coincides with the interest compounding period is called simple annuity, where as an annuity in which the payment period differs from the interest compounding period is called a complex annuity. The annuities considered in this study have terms given by fixed time intervals, periodic payments equal in size, payments made at the end of the period, and payment periods coincide with the interest compounding periods.

3.1 Future Value of Annuity

Future value of annuity is the sum of compound amount of payments made each accumulating to the end of the term (Awogbemi and Ojo, 2012).

The future value of an annuity of n payments paid at the end of each investment period into an account that earns interest at the rate of *i* per period is

$$F_v = m\left[\frac{(1+i)^N - 1}{i}\right] \qquad (16)$$

Let the periodic payments of an annuity be denoted by m, the total number of compounding periods in t years by N = nt and the number of compounding periods in a year by n. Then the future value to which the deposit would have grown at the time of N^{th} deposit is

$$\begin{aligned} F_v &= m + m(1+i) + m(1+i)^2 + \cdots + m(1+i)^{N-1} \\ &= m\frac{r^n - 1}{r-1}, \quad r > 1 \\ &= m\frac{(1+i)^N - 1}{1+i-1} \\ &= m\frac{(1+i)^N - 1}{i} \qquad (17) \end{aligned}$$

3.2 Present Value of Annuity

The present value of an annuity is the amount of money at hand that is equivalent to a series of equal payments in the future. The interest is in depositing lump sum that will have the same value as the annuity at the end of some time period (Kekere and Awogbemi, 2010).

Setting the future value based on compound interest to future value of annuity, we have

$$\begin{aligned} P_v(1+i)^N &= m\left[\frac{(1+i)^N - 1}{i}\right] \\ P_v &= \frac{m}{i}\left[\frac{(1+i)^N - 1}{(1+i)^N}\right] \\ &= \frac{m}{i}\left[1 - \frac{1}{(1+i)^N}\right] \\ &= m\left[\frac{1-(1+i)^{-N}}{i}\right] \qquad (18) \end{aligned}$$

If a loan is amortized, then the present value of annuity is used to generate the repayment of periodic equal installments m by algebraically solving for it in P_v as

$$m = \frac{(P_v)i}{1-(1+i)^{-N}} \qquad (19)$$

3.3 Loan Repayment

Suppose a loan of amount P is to be repaid discretely in n times per year over t years. The unpaid portion of the loan is charged interest at an annual rate r compounded n times per year. The discrete payment m can be obtained. The present value of all the payments should equal the amount borrowed.

If the payment is made at the end of the first compounding period, then the present value of all the payments denoted by P is

$$P = m\left(1+\frac{r}{n}\right)^{-1} + m\left(1+\frac{r}{n}\right)^{2} + \cdots + m\left(1+\frac{r}{n}\right)^{-nt}$$

$$= m\left(1+\frac{r}{n}\right)^{-1}\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{1-\left(1+\frac{r}{n}\right)^{-1}}\right]$$

$$= m\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}$$

$$= m\frac{n}{r}\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]$$

$$\Rightarrow m = P\frac{r}{n}\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]^{-1} \qquad (20)$$

3.4 Mortgage Loan

Suppose a mortgage loan is secured in the amount of L, with an n equal monthly payments of amount m, where the annual interest rate, r is compounded monthly. Setting the sum of the present values of all the payments to the amount loaned implies that

$$L = \sum_{i=1}^{n} \frac{m}{\left(1+\frac{r}{12}\right)^{i}} \;, \; i \text{ is rate per period}$$

$$= m\sum_{i=1}^{n}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\sum_{i=0}^{n-1}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\left[\frac{1-\left(1-\frac{r}{12}\right)^{-n}}{1-\left(1+\frac{r}{12}\right)^{-1}}\right]$$

$$= m\frac{\left[1-\left(1+\frac{r}{12}\right)^{-n}\right]}{\left(1+\frac{r}{12}\right)^{-1}-1}$$

$$= \frac{12m}{r}\left[1-\left(1+\frac{r}{12}\right)^{-n}\right] \tag{21}$$

Expressing m as a function of L, r and n, we have $m = L\frac{r}{12}\left[1-\left(1+\frac{r}{12}\right)^{-n}\right]^{-1}$

If L_k denotes the outstanding balance immediately after the k^{th} payments, then L_k is the sum of the present values of the remaining payments

$$L_k = \sum_{i=1}^{n-k}\frac{m}{\left(1+\frac{r}{12}\right)^{i}}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\sum_{i=0}^{n-k-1}\left(1+\frac{r}{12}\right)^{-i}$$

$$= m\left(1+\frac{r}{12}\right)^{-1}\frac{1-\left(1+\frac{r}{12}\right)^{-n+k}}{1-\left(1+\frac{r}{12}\right)^{-1}}$$

$$= \frac{m\left[1-\left(1+\frac{r}{12}\right)^{-n+k}\right]}{\left(1+\frac{r}{12}\right)^{-1}-1}$$

$$= \frac{12m}{r}\left[1-\left(1+\frac{r}{12}\right)^{-n+k}\right] \tag{22}$$

The amount of interest in the k^{th} payment denoted by

$$P_k = L_{k-1}\left(\frac{r}{12}\right)$$

$$= m\left[1-\left(1+\frac{r}{12}\right)^{-n+k-1}\right] \tag{23}$$

The repaid amount in the k^{th} repayment denoted by

$$R_k = m - P_k$$

$$= m\left(1+\frac{r}{12}\right)^{-n+k-1} \tag{24}$$

4. Conclusion

Interest rate theory has been employed as a veritable tool in this research work to generate the future and present values of annuity, the results of which are applied in repayment of loans and mortgage loans. The effective rate of interest, which is useful in comparing alternative investment opportunities, depends on the nominal rate and the conversion periods.

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Declaration of Conflict

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